

Construction of bipotentials and a minimax theorem of Fan

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Abstract

The bipotential theory is based on an extension of Fenchel's inequality, with several powerful applications related to non associated constitutive laws in Mechanics: frictional contact [12], non-associated Drucker-Prager model [1], or Lemaitre plastic ductile damage law [2], to cite a few.

This is a second paper on the mathematics of the bipotentials, following [4]. We prove here another reconstruction theorem for a bipotential from a convex lagrangian cover, this time using a convexity notion related to a minimax theorem of Fan.

Key words: bipotentials, minimax theorems

MSC-class: 49J53; 49J52; 26B25

1 Introduction

In Mechanics, the theory of standard materials is a well-known application of Convex Analysis. However, the so-called non-associated constitutive laws cannot be cast in the mould of the standard materials.

From the mathematical viewpoint, a non associated constitutive law is a multivalued operator $T : X \rightarrow 2^Y$ which is not supposed to be monotone. Here X, Y are dual locally convex spaces, with duality product $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{R}$.

A possible way to study non-associated constitutive laws by using Convex Analysis, proposed first in [12], consists in constructing a "bipotential" function b of two variables, which physically represents the dissipation.

A bipotential function b is bi-convex, satisfies an inequality generalizing Fenchel's one, $\forall x \in X, y \in Y, b(x, y) \geq \langle x, y \rangle$, and a relation involving partial subdifferentials of b with respect to variables x, y . In the case of associated constitutive laws the bipotential has the expression $b(x, y) = \phi(x) + \phi^*(y)$.

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The graph of a bipotential b is simply the set $M(b) \subset X \times Y$ of those pairs (x, y) such that $b(x, y) = \langle x, y \rangle$. A multivalued operator $T : X \rightarrow 2^Y$ is expressed with the help of the bipotential b if the graph of T (in the usual sense) equals $M(b)$.

The non associated constitutive laws which can be expressed with the help of bipotentials are called in Mechanics implicit, or weak, normality rules. They have the form of an implicit relation between dual variables, $y \in \partial b(\cdot, y)(x)$.

Among the applications of bipotentials to Solid Mechanics we cite: Coulomb's friction law [9], non-associated Drucker-Prager [11] and Cam-Clay models [10] in Soil Mechanics, cyclic Plasticity ([9],[3]) and Viscoplasticity [6] of metals with non linear kinematical hardening rule, Lemaitre's damage law [2], the coaxial laws ([5],[13]). A review of these laws expressed in terms of bipotentials can be found in [5] and [13].

In order to better understand the bipotential approach, in the paper [4] we solved two key problems: (a) when the graph of a given multivalued operator can be expressed as the set of critical points of a bipotentials, and (b) a method of construction of a bipotential associated (in the sense of point (a)) to a multivalued, typically non monotone, operator.

Our main tool was the notion of convex lagrangian cover of the graph of the multivalued operator, and a related notion of implicit convexity of this cover.

In this paper we prove another reconstruction theorem for a bipotential from a convex lagrangian cover, this time using a convexity notion related to a minimax theorem of Fan.

2 Notations and definitions

X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{R}$. The topologies of the spaces X, Y are compatible with the duality product, that is: any continuous linear functional on X (resp. Y) has the form $x \mapsto \langle x, y \rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y \rangle$, for some $x \in X$).

We use the notation: $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Given a function $\phi : X \rightarrow \bar{\mathbb{R}}$, the domain $dom \phi$ is the set of points with value other than $+\infty$. The polar of ϕ , or Fenchel conjugate, $\phi^* : Y \rightarrow \bar{\mathbb{R}}$ is defined by: $\phi^*(y) = \sup \{ \langle y, x \rangle - \phi(x) \mid x \in X \}$.

We denote by $\Gamma(X)$ the class of convex and lower semicontinuous functions $\phi : X \rightarrow \bar{\mathbb{R}}$. The class of convex and lower semicontinuous functions $\phi : X \rightarrow \mathbb{R}$ is denoted by $\Gamma_0(X)$.

The subdifferential of a function $\phi : X \rightarrow \bar{\mathbb{R}}$ in a point $x \in dom \phi$ is the (possibly empty) set:

$$\partial\phi(x) = \{u \in Y \mid \forall z \in X \langle z - x, u \rangle \leq \phi(z) - \phi(x)\} .$$

In a similar way is defined the subdifferential of a function $\psi : Y \rightarrow \bar{\mathbb{R}}$ in a point $y \in dom \psi$, as the set:

$$\partial\psi(y) = \{v \in X \mid \forall w \in Y \langle v, w - y \rangle \leq \psi(w) - \psi(y)\} .$$

With these notations we have the Fenchel inequality: let $\phi : X \rightarrow \bar{\mathbb{R}}$ be a convex lower semicontinuous function. Then:

- (i) for any $x \in X, y \in Y$ we have $\phi(x) + \phi^*(y) \geq \langle x, y \rangle$;
- (ii) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial\phi(x) \iff x \in \partial\phi^*(y) \iff \phi(x) + \phi^*(y) = \langle x, y \rangle .$$

Definition 2.1 To a graph $M \subset X \times Y$ we associate the multivalued operators:

$$X \ni x \mapsto m(x) = \{y \in Y \mid (x, y) \in M\} ,$$

$$Y \ni y \mapsto m^*(y) = \{x \in X \mid (x, y) \in M\} .$$

The domain of the graph M is by definition $\text{dom}(M) = \{x \in X \mid m(x) \neq \emptyset\}$. The image of the graph M is the set $\text{im}(M) = \{y \in Y \mid m^*(y) \neq \emptyset\}$.

3 Bipotentials

The notions and results in this section were introduced or proved in [4].

Definition 3.1 A bipotential is a function $b : X \times Y \rightarrow \bar{\mathbb{R}}$ with the properties:

- (a) b is convex and lower semicontinuous in each argument;
- (b) for any $x \in X, y \in Y$ we have $b(x, y) \geq \langle x, y \rangle$;
- (c) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y \rangle . \quad (3.0.1)$$

The graph of b is

$$M(b) = \{(x, y) \in X \times Y \mid b(x, y) = \langle x, y \rangle\} . \quad (3.0.2)$$

Examples. (1.) (Separable bipotential) If $\phi : X \rightarrow \bar{\mathbb{R}}$ is a convex, lower semicontinuous potential, consider the multivalued operator $\partial\phi$ (the subdifferential of ϕ). The graph of this operator is the set

$$M(\phi) = \{(x, y) \in X \times Y \mid \phi(x) + \phi^*(y) = \langle x, y \rangle\} . \quad (3.0.3)$$

$M(\phi)$ is *maximally cyclically monotone* [8] Theorem 24.8. Conversely, if M is closed and maximally cyclically monotone then there is a convex, lower semicontinuous ϕ such that $M = M(\phi)$.

To the function ϕ we associate the *separable bipotential*

$$b(x, y) = \phi(x) + \phi^*(y).$$

Indeed, the Fenchel inequality can be reformulated by saying that the function b , previously defined, is a bipotential. More precisely, the point (b) (resp. (c)) in the definition of a bipotential corresponds to (i) (resp. (ii)) from Fenchel inequality.

The bipotential b and the function ϕ have the same graph: $M(b) = M(\phi)$.

(2.) (Cauchy bipotential) Let $X = Y$ be a Hilbert space and let the duality product be equal to the scalar product. Then we define the *Cauchy bipotential* by the formula

$$b(x, y) = \|x\| \|y\|.$$

Let us check the Definition (3.1) The point (a) is obviously satisfied. The point (b) is true by the Cauchy-Schwarz-Bunyakovsky inequality. We have equality in the Cauchy-Schwarz-Bunyakovsky inequality $b(x, y) = \langle x, y \rangle$ if and only if there is $\lambda > 0$ such that $y = \lambda x$ or one of x and y vanishes. This is exactly the statement from the point (c), for the function b under study.

The graph $M(b)$ is the set of pairs of collinear and with same orientation vectors. It can not be expressed by a separable bipotential because $M(b)$ is not a cyclically monotone graph.

Definition 3.2 *The non empty set $M \subset X \times Y$ is a BB-graph (bi-convex, bi-closed) if for all $x \in \text{dom}(M)$ and for all $y \in \text{im}(M)$ the sets $m(x)$ and $m^*(y)$ are convex and closed.*

The following theorem gives a necessary and sufficient condition for the existence of a bipotential associated to a constitutive law M .

Theorem 3.3 *Given a non empty set $M \subset X \times Y$, there is a bipotential b such that $M = M(b)$ if and only if M is a BB-graph.*

Given the BB-graph M , the uniqueness of bipotential b such that $M = M(b)$ is not true. For example, in the case of the Cauchy bipotential b , the proof of theorem 3.3 (theorem ... [4]) provides a bipotential, denoted by b_∞ , such that $M(b) = M(b_\infty)$ but $b \neq b_\infty$. This is in contrast with the case of a maximal cyclically monotone graph M , when by Rockafellar theorem ([8] Theorem 24.8.) we have a method to reconstruct unambiguously the associated separable bipotential.

We noticed that in mechanical applications, we were able to reconstruct the physically relevant bipotentials b starting from $M(b)$, by knowing a little more than the graph $M(b)$. This supplementary information is encoded in the following notion.

Definition 3.4 *Let $M \subset X \times Y$ be a non empty set. A convex lagrangian cover of M is a function $\lambda \in \Lambda \mapsto \phi_\lambda$ from Λ with values in the set $\Gamma(X)$, with the properties:*

- (a) *The set Λ is a non empty compact topological space,*
- (b) *Let $f : \Lambda \times X \times Y \rightarrow \bar{\mathbb{R}}$ be the function defined by*

$$f(\lambda, x, y) = \phi_\lambda(x) + \phi_\lambda^*(y).$$

Then for any $x \in X$ and for any $y \in Y$ the functions $f(\cdot, x, \cdot) : \Lambda \times Y \rightarrow \bar{\mathbb{R}}$ and $f(\cdot, \cdot, y) : \Lambda \times X \rightarrow \bar{\mathbb{R}}$ are lower semi continuous on the product spaces $\Lambda \times Y$ and respectively $\Lambda \times X$ endowed with the standard topology,

(c) We have

$$M = \bigcup_{\lambda \in \Lambda} M(\phi_\lambda) \quad .$$

Not any BB-graph admits a convex lagrangian cover. There exist BB-graphs admitting only one convex lagrangian cover (up to reparametrization), as well as BB-graphs which have infinitely many lagrangian covers. The problem of describing the set of all convex lagrangian covers of a BB-graph seems to be difficult. We shall not discuss this problem here, but see the sections 5 and 8 in [4].

The results in this paper apply only to BB-graphs admitting at least one convex lagrangian cover.

To a convex lagrangian cover we associate a function which will turn out to be a bipotential, under some supplementary hypothesis.

Definition 3.5 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M . To the cover we associate the function $b : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ by the formula*

$$b(x, y) = \inf \{ \phi_\lambda(x) + \phi_\lambda^*(y) : \lambda \in \Lambda \} = \inf_{\lambda \in \Lambda} f(\lambda, x, y) \quad .$$

In [4] we imposed an implicit convexity inequality in order to get a function b which is a bipotential. We need two definitions.

Definition 3.6 *Let Λ be an arbitrary non empty set and V a real vector space. The function $f : \Lambda \times V \rightarrow \bar{\mathbb{R}}$ is implicitly convex if for any two elements $(\lambda_1, z_1), (\lambda_2, z_2) \in \Lambda \times V$ and for any two numbers $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ there exists $\lambda \in \Lambda$ such that*

$$f(\lambda, \alpha z_1 + \beta z_2) \leq \alpha f(\lambda_1, z_1) + \beta f(\lambda_2, z_2) \quad .$$

Definition 3.7 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M and $f : \Lambda \times X \times Y \rightarrow \mathbb{R}$ the associated function introduced in Definition 3.4, that is the function defined by*

$$f(\lambda, z, y) = \phi_\lambda(z) + \phi_\lambda^*(y) \quad .$$

The cover is bi-implicitly convex (or a BIC-cover) if for any $y \in Y$ and $x \in X$ the functions $f(\cdot, \cdot, y)$ and $f(\cdot, x, \cdot)$ are implicitly convex in the sense of Definition 3.6.

In the case of $M = M(\phi)$, with ϕ convex and lower semi continuous (this corresponds to separable bipotentials), the set Λ has only one element $\Lambda = \{\lambda\}$ and we have only one potential ϕ . The associated bipotential from Definition 3.5 is obviously

$$b(x, y) = \phi(x) + \phi^*(y) \quad .$$

This is a BIC-cover in a trivial way: the implicit convexity conditions are equivalent with the convexity of ϕ, ϕ^* respectively.

With this convexity condition we obtained in [4] the following result.

Theorem 3.8 *Let $\lambda \mapsto \phi_\lambda$ be a BIC-cover of the BB-graph M and $b : X \times Y \rightarrow R$ defined by*

$$b(x, y) = \inf \{ \phi_\lambda(x) + \phi_\lambda^*(y) \mid \lambda \in \Lambda \} . \quad (3.0.4)$$

Then b is a bipotential and $M = M(b)$.

4 Main result

For simplicity, in this section we shall work only with lower semi continuous convex functions ϕ with the property that ϕ and its Fenchel dual ϕ^* take values in \mathbb{R} .

We reproduce here the following definition of convexity (in a generalized sense), given by K. Fan [7] p. 42.

Definition 4.1 *Let X, Y be two arbitrary non empty sets. The function $f : X \times Y \rightarrow \mathbb{R}$ is convex on X in the sense of Fan if for any two elements $x_1, x_2 \in X$ and for any two numbers $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ there exists a $x \in X$ such that for all $y \in Y$:*

$$f(x, y) \leq \alpha f(x_1, y) + \beta f(x_2, y).$$

With the help of the previous definition we introduce a new convexity condition for a convex lagrangian cover.

Definition 4.2 *Let $\lambda \mapsto \phi_\lambda$ be a convex lagrangian cover of the BB-graph M . Consider the functions:*

$$g : X \times \Lambda \times X \rightarrow \mathbb{R} \quad , \quad h : Y \times \Lambda \times Y \quad ,$$

given by $g(x, \lambda, z) = \phi_\lambda(x) - \phi_\lambda(z)$, respectively $h(y, \lambda, u) = \phi_\lambda^(y) - \phi_\lambda^*(u)$.*

The cover is Fan bi-implicitly convex if for any $x \in X, y \in Y$, the functions $g(x, \cdot, \cdot)$, $h(y, \cdot, \cdot)$ are convex in the sense of Fan on $\Lambda \times X, \Lambda \times Y$ respectively.

Recall the following minimax theorem of Fan [7], Theorem 2. In the formulation of the theorem words "convex" and "concave" have the meaning given in definition 4.1 (more precisely f is concave if $-f$ is convex in the sense of the before mentioned definition).

Theorem 4.3 (Fan) *Let X be a compact Hausdorff space and Y an arbitrary set. Let f be a real valued function on $X \times Y$ such that, for every $y \in Y$, $f(\cdot, y)$ is lower semicontinuous on X . If f is convex on X and concave on Y , then the expressions $\min_{x \in X} \sup_{y \in Y} f(x, y)$ and $\sup_{y \in Y} \min_{x \in X} f(x, y)$ have meaning, and*

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y) \quad .$$

The difficulty of theorem 3.8 boils down to the fact the class of convex functions is not closed with respect to the inf operator. Nevertheless, by using Fan theorem 4.3 we get the following general result.

Theorem 4.4 *Let Λ be a compact Hausdorff space and $\lambda \mapsto \phi_\lambda \in \Gamma_0(X)$ be a convex lagrangian cover of the BB-graph M such that:*

- (a) *for any $x \in X$ and for any $y \in Y$ the functions $\Lambda \ni \lambda \mapsto \phi_\lambda(x) \in \mathbb{R}$ and $\Lambda \ni \lambda \mapsto \phi_\lambda^*(y) \in \mathbb{R}$ are continuous,*
- (b) *the cover is Fan bi-implicitly convex in the sense of definition 4.2.*

Then the function $b : X \times Y \rightarrow \mathbb{R}$ defined by

$$b(x, y) = \inf \{ \phi_\lambda(x) + \phi_\lambda^*(y) \mid \lambda \in \Lambda \}$$

is a bipotential and $M = M(b)$.

Proof. For some of the details of the proof we refer to the proof of theorem 3.8 in [4] (in that paper theorem 4.12). There are five steps in that proof. In order to prove our theorem we have only to modify the first two steps: we want to show that for any $x \in \text{dom}(M)$ and any $y \in \text{im}(M)$ the functions $b(\cdot, y)$ and $b(x, \cdot)$ are convex and lower semi continuous.

For $(x, y) \in X \times Y$ let us define the function $\overline{xy} : \Lambda \times X \rightarrow \mathbb{R}$ by

$$\overline{xy}(\lambda, z) = \langle z, y \rangle + \phi_\lambda(x) - \phi_\lambda(z) \quad .$$

We check now that \overline{xy} verifies the hypothesis of theorem 4.3. Indeed, the hypothesis (a) implies that for any $z \in X$ the function $\overline{xy}(\cdot, z)$ is continuous. Notice that

$$\overline{xy}(\lambda, z) = \langle z, y \rangle + g(x, \lambda, z) \quad .$$

It follows from hypothesis (b) that the function \overline{xy} is convex on Λ in the sense of Fan.

In order to prove the concavity of \overline{xy} on X , it suffices to show that for any $z_1, z_2 \in X$, for any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$, we have the inequality

$$\overline{xy}(\lambda, \alpha z_1 + \beta z_2) \leq \alpha \overline{xy}(\lambda, z_1) + \beta \overline{xy}(\lambda, z_2)$$

for any $\lambda \in \Lambda$. This inequality is equivalent with

$$\langle \alpha z_1 + \beta z_2, y \rangle - \phi_\lambda(\alpha z_1 + \beta z_2) \leq \alpha (\langle z_1, y \rangle - \phi_\lambda(z_1)) + \beta (\langle z_2, y \rangle - \phi_\lambda(z_2))$$

for any $\lambda \in \Lambda$. But this is implied by the convexity of ϕ_λ for any $\lambda \in \Lambda$.

In conclusion the function \overline{xy} satisfies the hypothesis of theorem 4.3. We deduce that

$$\min_{\lambda \in \Lambda} \sup_{z \in X} \overline{xy}(\lambda, z) = \sup_{z \in X} \min_{\lambda \in \Lambda} \overline{xy}(\lambda, z) \quad .$$

Let us compute the two sides of this equality.

For the left hand side (LHS) we have:

$$\begin{aligned}
LHS &= \min_{\lambda \in \Lambda} \sup_{z \in X} \{ \langle z, y \rangle + \phi_\lambda(x) - \phi_\lambda(z) \} = \\
&= \min_{\lambda \in \Lambda} \left\{ \phi_\lambda(x) + \sup_{z \in X} \{ \langle z, y \rangle - \phi_\lambda(z) \} \right\} = \\
&= \min_{\lambda \in \Lambda} \{ \phi_\lambda(x) + \phi_\lambda^*(y) \} = b(x, y) \quad .
\end{aligned}$$

For the right hand side (RHS) we have:

$$\begin{aligned}
RHS &= \sup_{z \in X} \min_{\lambda \in \Lambda} \{ \langle z, y \rangle + \phi_\lambda(x) - \phi_\lambda(z) \} = \\
&= \sup_{z \in X} \left\{ \langle z, y \rangle - \max_{\lambda \in \Lambda} \{ \phi_\lambda(z) - \phi_\lambda(x) \} \right\} \quad .
\end{aligned}$$

Let $\bar{x} : X \rightarrow \mathbb{R}$ be the function

$$\bar{x}(z) = \max_{\lambda \in \Lambda} \{ \phi_\lambda(z) - \phi_\lambda(x) \} \quad .$$

Then the right hand side RHS is in fact:

$$RHS = \bar{x}^*(y) \quad .$$

Therefore we proved the equality:

$$b(x, y) = \bar{x}^*(y) \quad .$$

This shows that the function b is convex and lower semicontinuous in the second argument.

In order to prove that b is convex and lower semicontinuous in the first argument, replace ϕ_λ by ϕ_λ^* in the previous reasoning. ■

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